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INVARIANT INTEGRALS IN THE LEONOV–PANASYUK–DUGDALE MODEL OF A CRACK

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In this paper, we calculate some ordinary invariant integrals and higher-order integrals that contain derivatives of displacement and stress fields. The arc of integration ends on the crack sides and encloses the crack tip together with the tip zone. Values of the integrals are used to determine the parameters of the end zone — its length and the intensity of cohesion forces — (the classical integrals $J_1 = J$ and $J_r = M$ are related only to energy characteristics). An approach to the calculation of invariant integrals using their common properties and eliminating cumbersome manipulations is described.

1. Formulation of the Problem. Let Ω be a plane isotropic body with cracks whose sides are included in the boundary $\partial\Omega$. We fix a straight crack M and relate it to Cartesian coordinates so that

$$M = \{x : x_2 = 0, \quad x_1 \in (-a, 0)\}. \tag{1.1}$$

Assume that Ξ is a vicinity of the point $x = 0$, Γ is a simple contour that connects the sides M^+ and M^- of the crack (1.1) inside $\Xi \backslash M$, and mass forces are absent in Ξ. The crack sides are free of stresses. According to the Leonov–Panasyuk–Dugdale model [1, 2], in the mouth $M_l = \{x \in M : x_1 \geqslant -l\}$ act cohesion forces with intensity q, and $M_l \subset \Xi$. In other words, the displacement vector $u = (u_1, u_2)$ satisfies the relations

$$L(\nabla)u(x) \equiv \mu\nabla \cdot \nabla u(x) + (\lambda + \mu)\nabla\nabla \cdot u(x) = 0, \quad x \in \Xi\backslash M; \tag{1.2}$$

$$\sigma_{12}(u; x) = 0, \quad x \in M^{\pm} \cap \Xi; \tag{1.3}$$

$$\sigma_{22}(u; x) = 0, \quad x \in M^{\pm} \cap \Xi, \quad x_1 < -l; \tag{1.4}$$

$$\sigma_{22}(u; x) = q, \quad x \in M^{\pm} \cap \Xi, \quad x_1 > -l. \tag{1.5}$$

Here $\nabla = \text{grad}$, the dot denote scalar products (i.e., $\nabla \cdot \nabla = \Delta$ is the Laplacian and $\nabla \cdot = \text{div}$), λ and μ are the Lamé coefficients, and $\sigma_{ij}(u)$ are Cartesian components of the stress tensor $\sigma(u)$:

$$\sigma_{ij}(u) = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \lambda\delta_{i,j}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\right), \quad i, j = 1, 2. \tag{1.6}$$

As usual, the length $l > 0$ of the zone of action of the cohesion force is set (automatically, according to the postulates of the adopted model) such that the stresses (1.6) remain limited up to the crack tip, i.e., the stress intensity coefficients vanish. The mouth of the crack is small ($l \ll a$), and we further assume that the ends of the arc Γ do not belong to M_l.

The goal of this paper is to determine all possible characteristics of the mouth of the crack from values of various invariant integrals calculated on the contour Γ away from the apex O by solving problem (1.2)–(1.5).

2. First-Order Invariant Integrals. Cherepanov [3] and Rice [4] proposed the path-independent integral

$$J_1(u; \Gamma) = \int\limits_{\Gamma} \left[W(u; x)\cos(n, x_1) - \frac{\partial u}{\partial x_1}(x) \cdot \sigma^{(n)}(u; x)\right] ds, \tag{2.1}$$

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where ds is an element of the arc length, $n = (n_1, n_2)$ is the unit normal vector (external with respect to the region bounded by Γ), $n_1 = \cos(n, x_1)$ is its directing cosine, $\sigma^{(n)}$ is the vector of normal stresses with components $\sigma_i^{(n)} = n_1\sigma_{1i} + n_2\sigma_{2i}$, $W(u)$ is the density of the elastic-energy integral, $W(u;x) = W(u,u;x)$, and

$$W(u,v;x) = \frac{1}{2\mu}\sum_{i,j=1}^{2}\Big[\sigma_{ij}(u;x)\sigma_{ij}(v;x) - \frac{\lambda}{2(\lambda+\mu)}\sigma_{ii}(u;x)\,\sigma_{jj}(v;x)\Big]. \tag{2.2}$$

Knowles and Sternberg [5] and Budiansky and Rice [6] proposed another invariant integral:

$$J_r(u;\Gamma) = \int_\Gamma \Big\{W(u;x)x\cdot n(x) - \Big(x_1\frac{\partial u}{\partial x_1}(x) + x_2\frac{\partial u}{\partial x_2}(x)\Big)\cdot\sigma^{(n)}(u;x)\Big\}\,ds. \tag{2.3}$$

It is known that [with satisfaction of Eq. (1.2) and the boundary conditions (1.3) and (1.4) near Γ], integrals (2.1) and (2.3) coincide, respectively, with the integrals

$$I_1(u;\Gamma) = \frac{1}{2}\int_\Gamma \Big\{u(x)\cdot\sigma^{(n)}\Big(\frac{\partial u}{\partial x_1};x\Big) - \frac{\partial u}{\partial x_1}(x)\cdot\sigma^{(n)}(u;x)\Big\}\,ds; \tag{2.4}$$

$$I_r(u;\Gamma) = \frac{1}{2}\int_\Gamma \Big\{u(x)\cdot\sigma^{(n)}(Du;x) - Du(x)\cdot\sigma^{(n)}(u;x)\Big\}\,ds. \tag{2.5}$$

In (2.5), D is a scalar differential operator:

$$D = x\cdot\nabla = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}. \tag{2.6}$$

Note that, under the above conditions, the invariance of integrals (2.4) and (2.5) is beyond question: it suffices to recall the Betti identity (the Green formula) and note that, by virtue of (1.2)–(1.4),

$$\begin{gathered} L(\nabla)\frac{\partial u}{\partial x_1} = 0 \text{ on } \Xi\backslash M, \qquad \sigma_{i2}\Big(\frac{\partial u}{\partial x_1}\Big) = 0 \text{ on } (M^\pm\backslash M_l^\pm)\cap\Xi, \\ L(\nabla)Du = (D+2)\,L(\nabla)u \text{ on } \Xi\backslash M, \\ \sigma_{i2}(Du) = (D+1)\,\sigma_{i2}(u) = \Big(x_1\frac{\partial}{\partial x_1} + 1\Big)\sigma_{i2}(u) \text{ on } (M^\pm\backslash M_l^\pm)\cap\Xi, \qquad i = 1, 2. \end{gathered} \tag{2.7}$$

For integrals (2.1) and (2.4), the above-mentioned fact is verified, for example, in [7] and [8, § 7.3]. Not having an exact reference, we establish here the coincidence of integrals (2.2) and (2.5). We assume that χ is a cutting function that is smooth in $\Xi\backslash M$, equal to unity near Γ, and vanishes where relations (1.2)–(1.4) are violated. Π is a region located inside Γ and bounded by the arc Γ, another similar arc Γ' (on it, $\chi = 0$), and by segments $m^\pm$ of the sides $M^\pm$. We set $m = m^+\cup m^-$. Integrating more than once by parts, we obtain

$$2I_r(u;\Gamma) = 2I_r(\chi u;\Gamma) = \int_\Pi \{\chi u\cdot LD(\chi u) - D(\chi u)\cdot L(\chi u)\}dx$$

$$-\int_m \Big\{\chi u\cdot\sigma^{(n)}(D(\chi u)) - D(\chi u)\cdot\sigma^{(n)}(\chi u)\Big\}ds = \int_\Pi \{\chi u\cdot(D+2)\,L(\chi u) - D(\chi u)\cdot L(\chi u)\}\,dx$$

$$-\int_m \Big\{\chi u\cdot\Big(x_1\frac{\partial}{\partial x_1}+1\Big)\sigma^{(n)}(\chi u) - x_1\frac{\partial}{\partial x_1}(\chi u)\cdot\sigma^{(n)}(\chi u)\Big\}dx_1$$

$$= -2\int_\Pi D(\chi u)\cdot L(\chi u)\,dx + 2\int_m x_1\frac{\partial}{\partial x_1}(\chi u)\cdot\sigma^{(n)}(\chi u)\,dx_1$$

$$= 4\int_\Pi W(D(\chi u),\chi u)\,dx - 2\int_\Gamma Du\cdot\sigma^{(n)}(u)\,ds = 2\int_\Pi (D+2)\,W(\chi u,\chi u)\,dx - 2\int_\Gamma Du\cdot\sigma^{(n)}(u)\,ds$$

$$= 2\int_\Gamma (x\cdot n)\, W(u,u)\, dx - 2\int_\Gamma Du\cdot \sigma^{(n)}(u)\, ds = 2J_r(u;\Gamma). \tag{2.8}$$

In these transformations, we took into account equalities (2.7) and the fact that $L(\chi u) = 0$ on $\Gamma\cup\Gamma'$ and $\sigma^{(n)}(\chi u) = 0$ at the ends of the segments $m^\pm$. In addition, on m, we use the relation $x\cdot n = 0$, which implies that, with "transfer" of D, no integrals occur on the sides $M^\pm$.

We calculate integrals (2.1) and (2.4). We close the arc Γ by the segments γ^+ and γ^- of the crack sides which end at the tip $x = 0$. Since, at the tip, stresses do not have singularities, by virtue of the invariance of the integrals, we have

$$J_1(u;\Gamma) = I_1(u;\Gamma) = -I_1(u;\gamma^+) - I_1(u;\gamma^-) = -\frac{1}{2}\int_{\gamma^+\cup\gamma^-}\Big\{u(x)\cdot\sigma^{(n)}\Big(\frac{\partial u}{\partial x_1};x\Big) - \frac{\partial u}{\partial x_1}(x)\cdot\sigma^{(n)}(u;x)\Big\}dx. \tag{2.9}$$

With closure of the arc Γ, we assume that on the new portion of the contour, the direction of the normal is inherited. Therefore, after the second sign, the minus sign appears in equality (2.9). We use this convention throughout the paper.

It appears at first glance that, by virtue of (1.3)–(1.5), the minuend in braces in (2.9) vanishes, and the right side of (2.9) coincides with the relation

$$\frac{1}{2}\sum_{\pm}\mp\int_{-l}^{0}\frac{\partial u_2}{\partial x_1}(x_1,\pm 0)\, q\, dx_1 = \frac{q}{2}[u_2(-l,+0) - u_2(-l,-0)].$$

But this is faulty, because, due to the jump of the function $x_1\mapsto \sigma_{22}(\partial u/\partial x_1; x_1,\pm 0)$ at the point $x_1 = -l$, the quantity $\sigma_{22}(\partial u/\partial x_1; x_1,\pm 0)$ includes the generalized function $q\delta(x_1 + l)$, which is proportional to the Dirac δ-function. As a result, taking into account the omitted singular term, we find

$$J_1(u;\Gamma) = I_1(u;\Gamma) = q[u_2(-l,+0) - u_2(-l,-0)]. \tag{2.10}$$

From the same reasoning and by virtue of equalities (2.7), we obtain the following chain of relations for integrals (2.3) and (2.5):

$$J_r(u;\Gamma) = I_r(u;\Gamma) = -I_r(u;\gamma^+) - I_r(u;\gamma^-) = -\frac{1}{2}\int_{\gamma^+\cup\gamma^-}\Big\{u(x)\cdot(D+1)\sigma^{(n)}(u;x) - Du(x)\cdot\sigma^{(n)}(u;x)\Big\}ds$$

$$= -\frac{1}{2}\sum_{\pm}\pm q\Big\{u_2(-l,\pm 0)\, l + \int_{-l}^{0}\Big[u_2(x_1,\pm 0) - x_1\frac{\partial u_2}{\partial x_1}(x_1,\pm 0)\Big]dx_1\Big\} =$$

$$= -\sum_{\pm}\pm q\Big\{u_2(-l,\pm 0)\, l + \int_{-l}^{0} u_2(x_1,\pm 0)\, dx_1\Big\}. \tag{2.11}$$

A rigorous proof of formulas (2.10) and (2.11) using analysis of the displacement fields near the points $Q^\pm = (-l,\pm 0)$ is given in Sec. 3.

The quantity (2.10) coincides with the energy-release rate s under displacement of the mouth edge due to crack extension or decrease in load (see, for example, [9]). The sum of the latter integrals for $x_1\in(-l,0)$ from (2.11) is the work $A(u)$ of cohesion forces on displacements u. This corresponds to the general concept of invariant integrals (see [3–6] etc.; probably, formulas (2.10) and (2.11) are known, but we do not know exact references). However, apart from $A(u)$, (2.11) includes the term $-ls$ [cf. with (2.10)], which can be interpreted as the energy expended on the formation of the mouth. Thus,

$$A(u) = J_r(u;\Gamma) + lJ_1(u;\Gamma). \tag{2.12}$$

Equality (2.12) indicates the necessity of calculating the parameter l.

3. Asymptotic Representations of Displacements Near $Q^\pm$. In Sec. 2, we handled somewhat freely the generalized functions concentrated in $Q^\pm$. To perform more rigorous manipulations, we first elucidate

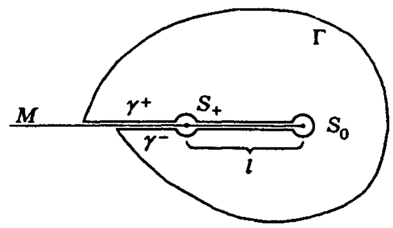

Fig. 1

the asymptotic behavior of the solution u near $Q^{\pm}$, replace integration along the segments $\gamma^{\pm}$ by integration along the arcs $\gamma_\delta^{\pm}$ shown in Fig. 1, and then find the limit as $\delta \to +0$ [in this section, the circle with the center 0 is not used; see further the comment on (5.5)].

Let $y = (y_1, y_2)$ and (ρ, θ) be Cartesian and polar coordinates with the center Q^+,

$$y_1 = x_1 + l = \rho\cos\theta, \qquad y_2 = x_2 = \rho\sin\theta. \tag{3.1}$$

We consider a displacement field w with the components

$$\begin{gathered} w_1(y) = (4\mu\pi)^{-1} q\{(1-æ)y_1\theta - (1+æ)y_2\ln\rho - y_2 - 2\pi y_1\}, \\ w_2(y) = (4\mu\pi)^{-1} q\{(1-æ)y_2\theta + (1+æ)y_1\ln\rho - y_1 + 2\pi y_2\} \\ \Big(æ = 3 - 4\nu = (\lambda+3\mu)(\lambda+\mu)^{-1}, \quad \nu = \lambda[2(\lambda+\mu)]^{-1}\Big). \end{gathered} \tag{3.2}$$

Direct calculations show that

$$\sigma_{12}(w; y_1, +0) = 0, \quad y_1 \in (-\infty, +\infty); \quad \sigma_{22}(w; y_1, +0) = \begin{cases} 0, & y_1 \in (-\infty, 0), \\ q, & y_1 \in (0, +\infty). \end{cases} \tag{3.3}$$

Therefore, by means of the field w from (3.2), one can compensate the jump on the right side of boundary conditions (1.4) and (1.5), and the difference $R = u - w$ becomes smooth near Q^+ and can be expanded in a Taylor series:

$$R(y) = u(x) - w(y) = R^0(y) + R^1(y) + R^2(y) + O(\rho^3), \quad \rho \to 0. \tag{3.4}$$

Here R^k is a vector polynomial that satisfies the homogeneous Lamé system and the conditions $\sigma_{i2}(R^k; y_1, 0) = 0$, $i = 1$ and 2. Furthermore,

$$\begin{aligned} \partial_1 w_1(y) &= (4\mu\pi)^{-1} q\Big\{(1-æ)\,\theta - 2y_1y_2\rho^{-2} - 2\pi\Big\}, \\ \partial_1 w_2(y) &= (4\mu\pi)^{-1} q\Big\{(1+æ)\,\ln\rho - 2y_2^2\rho^{-2} + æ\Big\}. \end{aligned} \tag{3.5}$$

Thus, the field $\partial_1 w$ differs from the classical solution of the problem of loading of a half plane by a normal concentrated force only by rigid translation (see, for example, [10, § 10.9]).

We now consider the integral along a semicircle with radius δ, obtained under integration along γ_δ^+:

$$A(X, Y) = \delta \int_0^\pi \Big\{Y(y)\cdot\sigma^{(\rho)}(X; y) - X(y)\cdot\sigma^{(\rho)}(Y; y)\Big\}\Big|_{\rho=\delta} d\theta; \tag{3.6}$$

$$\sigma_i^{(\rho)}(X) = \sigma_{i1}(X)\cos\theta + \sigma_{i2}(X)\sin\theta, \qquad i = 1, 2. \tag{3.7}$$

Let $X = \partial_1 w$ and $Y = e^1$ or $Y = e^2$, where e^i is the unit vector of the x_i axis. Substituting the stresses calculated according to (1.6), (3.7), and (3.5) into (3.6), we find

$$A(\partial_1 w, e^2) = q, \qquad A(\partial_1 w, e^1) = 0. \tag{3.8}$$

Equalities (3.8) support the fact noted after (3.5).

All manipulations are true for the point Q^-. Therefore, using the invariance of (2.1) and (2.4), we integrate along $\gamma_\delta^\pm$ (note the necessity of following the direction of the normal) and then δ let tend to zero. As a result, we obtain equalities (2.10). The same reasoning was used in the formal transformations (2.11) (see also Sec. 4, which describes methods used to facilitate calculations).

4. Refining the Asymptotic Formulas. In Ξ, using the displacement field u, we construct two elastic fields:

$$U(x) = \frac{\partial u}{\partial x_1}(x), \qquad V(x) = Du(x) - u(x). \tag{4.1}$$

By virtue of (1.2)–(1.5) and (2.7), U and V satisfy the homogeneous system (1.2), and the stresses $\sigma_{i2}(U)$ and $\sigma_{i2}(V)$ are eliminated on $M^\pm$ everywhere except maybe at the points $Q^\pm$. As previously, we restrict our consideration to the upper side $M^\pm$. The asymptotic representation for U follows immediately from (3.4):

$$U(x) = \partial_1 w(y) + \partial_1 R^1(y) + \partial_2 R^2(y) + O(\rho^2), \qquad \rho \to 0. \tag{4.2}$$

To write a similar formula for V, we transform to the coordinates (3.1) in (2.6) and obtain

$$D = x \cdot \nabla_x = -l\partial_1 + y \cdot \nabla_y = -l\partial_1 + D^{(l)}. \tag{4.3}$$

Clearly, $D^{(l)} R_k = kR_k$. In addition, from (3.2) we derive

$$D^{(l)} w(y) = w(y) + \omega(y), \quad \omega(y) = Nq(y_1 e^2 - y_2 e^1), \quad N = (1 + æ)/(4\pi\mu). \tag{4.4}$$

Thus, by virtue of (4.1), (4.3), and (4.4), we have

$$\begin{gathered} V(x) = -l\partial_1 w(y) + T^0(y) + T^1(y) + O(\rho^2), \quad \rho \to 0, \\ T^0(y) = -l\partial_1 R^1(y), \quad T^1(y) = -l\partial_1 R^2(y) + \omega(y), \end{gathered} \tag{4.5}$$

where $T^{(k)}$ is a vector polynomial of the power k.

Formulas similar to (4.2), (4.4), and (4.5) can also be written for $x_2 < 0$, but the polynomials R^k and T^k will, generally speaking, be different. To avoid cumbersome notation, we do not supply R^k, T^k, and w with the subscripts $\pm$ that distinguish the crack sides. Instead, we write $R^k(x_1, \pm 0)$ to distinguish traces of the polynomials on $M^\pm$ (in other words, R^k and T^k denote piecewise-polynomial vector functions).

We calculate the integrals $J_1(U;\Gamma)$, $J_1(V;\Gamma)$, etc., which, according to [11], will be called higher-order invariant integrals (they include derivatives of the field u). To simplify manipulations, we note some properties of the form A from (3.6); it is assumed that everywhere X and Y are solutions of the homogeneous problem in the half-space (w is not such, but $\partial_1 w$ and R^k and T^k fit). First, the form A is antisymmetric, i.e.,

$$A(X, Y) = -A(Y, X). \tag{4.6}$$

Second, by virtue of the Betti formula, relation (3.6) does not depend on ρ. Third, if $X(y) = \rho^\tau \Phi(\theta, \ln\rho)$ and $Y(y) = \rho^æ \Psi(\theta, \ln\rho)$ (the dependence on θ is smooth, the dependence on $\ln\rho$ is polynomial, and τ, $æ \in \mathbb{R}$), then $A(X, Y)$ is different from zero only if $\tau + æ = -1$ (this conclusion is obtained by passage to the limit $\delta \to +0$ or $\delta \to +\infty$ in the cases $\tau + æ > -1$ and $\tau + æ < -1$, respectively). Fourth,

$$A(\partial_1 X, Y) = -A(X, \partial_1 Y). \tag{4.7}$$

Formula (4.7) is established in [7] (see also [8, § 7.4]); it is obtained by "transferring" the derivative $\partial_1 = \partial/\partial y_1$ after transformation to a two-dimensional integral [cf. with (2.8)].

We find asymptotic relations for the fields u and U and V near the crack tip. Let (r, φ) be the polar coordinate, where $r = |x|$ and $\varphi \in (-\pi, \pi)$. Since we use the Leonov–Panasyuk–Dugdale model, the stress-intensity coefficients are equal to zero, and we have

$$u(x) = \Lambda(x) + r^{3/2}\big\{k_1 \Phi^{3,1}(\varphi) + k_2 \Phi^{3,2}(\varphi)\big\} + O(r^2). \tag{4.8}$$

In this case,

$$U(x) = \partial_1 \Lambda(0) + (3/2) r^{1/2}\big\{k_1 \Phi^{1,1}(\varphi) + k_2 \Phi^{1,2}(\varphi)\big\} + O(r^1); \tag{4.9}$$

$$V(x) = -\Lambda(0) + (1/2)r^{3/2}\big\{k_1\Phi^{3,1}(\varphi) + k_2\Phi^{3,2}(\varphi)\big\} + O(r^2). \tag{4.10}$$

Here Λ is a linear function, k_1 and k_2 are coefficients at the minor singularities of the solution, and $\Phi^{m,j}$ are vectors with following polar components

$$\Phi_r^{m,1}(\varphi) = (2\pi)^{-1/2}(4\mu m)^{-1}\Big\{(m-2)\cos\frac{1}{2}(m+2)\varphi + (2æ - m)\cos\frac{1}{2}(2-m)\varphi\Big\},$$

$$\Phi_\varphi^{m,1}(\varphi) = (2\pi)^{-1/2}(4\mu m)^{-1}\Big\{(-m+2)\sin\frac{1}{2}(m+2)\varphi - (2æ + m)\sin\frac{1}{2}(2-m)\varphi\Big\};$$

$$\Phi_r^{m,2}(\varphi) = (2\pi)^{-1/2}(4\mu m)^{-1}\Big\{(m+2)\sin\frac{1}{2}(m+2)\varphi - (2æ - m)\sin\frac{1}{2}(2-m)\varphi\Big\},$$

$$\Phi_\varphi^{m,2}(\varphi) = (2\pi)^{-1/2}(4\mu m)^{-1}\Big\{(m+2)\cos\frac{1}{2}(m+2)\varphi - (2æ + m)\cos\frac{1}{2}(2-m)\varphi\Big\}.$$

Note that (4.9) and (4.10) are obtained from (4.8) by simple differentiation. The second term in (4.9) is the same as that for a crack with completely free sides, i.e., $3k_1/2$ and $3k_2/2$ are the intensity-coefficients for the stresses generated by the displacement field U. Note that the model considered belongs to the theory of opening-mode cracks, in which $k_2 = 0$, but the presence of the second term in braces does not lead to additional difficulties, and, for generality (imaginary), we leave k_2.

5. Higher-Order Invariant Integrals. We calculate the integral $J_1(U;\Gamma)$, for which, by virtue of singularity of the stresses $\sigma_{ij}(U)$, the crack tip requires separate treatment. The result is known (see [3, 4] etc.): the integral $J_1(U;\cdot)$ calculated along the incomplete circle $S_0 = \{x : r = \delta, \varphi \in (-\pi, \pi)\}$ is equal to

$$\frac{1+æ}{4\mu}\left[\left(\frac{3}{2}k_1\right)^2 + \left(\frac{3}{2}k_2\right)^2\right] = \frac{9}{16}\,\frac{1+æ}{\mu}\,(k_1^2 + k_2^2). \tag{5.1}$$

We seek integrals along the semicircles $S_\pm = \{x : \rho = \delta, \pm\theta \in (0, \pi)\}$; for definiteness, we consider the upper semicircle. According to (2.4), (3.6), and (4.2), we have

$$-2J_1(U;S_+) = 2I_1(U;S_+) = A(\partial_1 U, U) = A(\partial_1^2 w + \partial_1^2 R^2, \partial_1 w + \partial_1 R^1 + \partial_1 R^2) + O(\delta), \quad \delta \to 0. \tag{5.2}$$

Owing to the second property of the form A, the residue $O(\delta)$ may not be written. Determining the orders of homogeneities for the vector functions on the right side of (5.2) and using the third property, we eliminate "superfluous" terms, and, by means of (4.6) and (4.7), we obtain

$$-2J_1(U;S_+) = A(\partial_1^2 w, \partial_1 R^2) - A(\partial_1 w, \partial_1^2 R^2) = -2A(\partial_1 w, \partial_1^2 R^2). \tag{5.3}$$

Since R^2 is a quadratic polynomial, taking into account (3.8), we have

$$A(\partial_1 w, \partial_1^2 R^2) = -q\partial_1^2 R_2^2(-l, +0). \tag{5.4}$$

Thus, replacing integration along Γ by integration along the arcs $S_\pm$ and S_0, and by the segments of the crack sides that connect the ends of these arcs [on these sections, the integral $J_1(U;\cdot)$ vanishes], according to (5.1), (5.3), and (5.4), we have

$$J_1(U;\Gamma) = \frac{9}{4}\pi N\,(k_1^2 + k_2^2) - q\mathbf{R}; \tag{5.5}$$

$$\mathbf{R} = \partial_1^2 R_2^2(-l, +0) - \partial_1^2 R_2^2(-l, -0). \tag{5.6}$$

The contribution of the crack tip to the remaining three integrals considered here is zero (because the corresponding integrands remain limited for $r \to 0$). The formulas

$$J_1(V;\Gamma) = -2lNq^2 + l^2 q\mathbf{R}; \tag{5.7}$$

$$J_r(U;\Gamma) = Nq^2 - lq\mathbf{R}; \tag{5.8}$$

$$J_r(V;\Gamma) = 3l^2 Nq^2 - l^3 q\mathbf{R} \tag{5.9}$$

are valid. We give manipulations that lead to these formulas. Proceeding as previously and invoking (4.4) and (4.5), for the case $J_1(V;\Gamma)$, we obtain the following chain of equalities:

$$\begin{aligned}-2J_1(V;S_+) &= A(-l\partial_1^2 w + \partial_1 T^1, -l\partial_1 w + T^0 + T^1) + O(\delta)\\ &= -lA(\partial_1^2 w, T^1) - lA(\partial_1 T^1, \partial_1 w) = 2lA(\partial_1 w, \partial_1 T^1)\\ &= 2lq\partial_1 T_2^1(-l,+0) = -2l^2 q\partial_1^2 R_2^2(-l,+0) + 2lNq^2.\end{aligned}$$

We calculate two other integrals. Note that, by virtue of (3.5) and (4.3), we have $D^{(l)}\partial_1 w(y) = Nqe^2$. Next, taking into account (3.8) and (4.3), and also (4.2) or (4.5), we have

$$\begin{aligned}-2J_r(U;S_+) &= A(-l\partial_1^2 w - l\partial_1^2 R^2 + D^{(l)}\partial_1 w + \partial_1 R^2, \partial_1 w + \partial_1 R^1 + \partial_1 R^2) + O(\delta)\\ &= -lA(\partial_1^2 w, \partial_1 R^2) - lA(\partial_1^2 R^2, \partial_1 w) + A(D^{(l)}\partial_1 w, \partial_1 w)\\ &= 2lA(\partial_1 w, \partial_1^2 R^2) - NqA(\partial_1 w, e^2) = 2lq\partial_1^2 R_2^2(-l,+0) - Nq^2,\end{aligned}$$

$$\begin{aligned}-2J_r(V;S_+) &= A(l^2\partial_1^2 w - l\partial_1 T^1 - lD^{(l)}\partial_1 w + T^1, -l\partial_1 w + T^0 + T^1) + O(\delta)\\ &= l^2 A(\partial_1^2 w, T^1) + l^2 A(\partial_1 T^1, \partial_1 w) + l^2 NqA(e^2, \partial_1 w)\\ &= -2l^2 A(\partial_1 w, \partial_1 T^1) - l^2 Nq^2 = 2l^3\partial_1^2 R_2^2(-l,+0) - 3l^2 Nq^2.\end{aligned}$$

Here we repeatedly used the properties of formula (3.6) indicated in Sec. 4. It remains to repeat the transformations for the lower semicircle S_- and perform the usual replacement of the path of integration.

6. Determining Characteristics of the Crack Mouth. Using (5.7)–(5.9), we obtain the following quadratic equation for the length:

$$l^2 J_r(U;\Gamma) + 2lJ_1(V;\Gamma) + J_r(V;\Gamma) = 0. \tag{6.1}$$

After determining l from (6.1), we can express the intensity of the cohesion forces by, for example, the equality

$$q = l^{-1}\{N^{-1}[J_r(V;\Gamma) + lJ_1(V;\Gamma)]\}^{1/2}. \tag{6.2}$$

Now formulas (2.10) and (6.2) yield the opening $u_2(-l,+0) - u_2(-l,-0)$ of the mouth of the crack. In addition, it is easy to find the work (2.12) of the cohesion forces. Note that, according to (4.3), we have

$$J_r(u;\Gamma) + lJ_1(u;\Gamma) = \frac{1}{2}\int_\Gamma \{u\cdot\sigma^{(n)}(D^{(l)}u) - D^{(l)}u\cdot\sigma^{(n)}(u)\}ds, \tag{6.3}$$

i.e., the sum (6.3) is the invariant integral (2.2) [or (2.5)] calculated in the y coordinates, related to the end of the crack mouth (it is not possible to avoid determining l). A combination similar to (6.3) also enters (6.2).

Using one of the equalities (5.7)–(5.9), one can find the difference (5.6) and then, by means of (5.5), the sum of the squares of the coefficients k_i from (4.8). However, we do not know the physical meaning of these quantities as characteristics of the crack mouth. Equally, the aforesaid is true for the coefficients of minor terms in series (4.8) and for the jumps of the polynomials R^3, R^4, ... in the Taylor formula (3.4) — they are all sought by iteration of the derivatives $\partial_1 = \partial/\partial x_1$ and $D = x\cdot\nabla$.

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